Lydia M. Bilinsky, SoMSS, Arizona State University, Tempe, AZ, USA
Steven M. Baer, SoMSS, Arizona State University, Tempe, AZ, USA

Slow passage through a Hopf bifurcation in spatially extended excitable systems: Some examples from neuroscience

Hopf bifurcation is a common mechanism by which a dynamical system featuring a constant parameter p has a critical value p_H, referred to as the Hopf point, such that for values of $p < p_H$ the system approaches a steady state, while for values of $p > p_H$ the system enters into sustained oscillations. It is known that when p is not constant in time, but rather is ramped up at a very slow rate from some initial $p_0 < p_H$, there is a delay in the onset of sustained oscillations: they do not ensue as soon as p exceeds p_H. The parameter value p_{crit} at which sustained oscillations do ensue for a given ramp depends on both the initial value of the ramp p_0 as well as its functional form. Several authors [2-3] have studied the case of a linear parameter ramp $p = p_0 + \epsilon t$, $\epsilon << 1$; Baer and Gaekel [1] have considered more general monotonic ramps, including accelerating ramps such as $p = p_0 + (\epsilon t)^2$ and deaccelerating ramps such as $p = p_0 + \sqrt{\epsilon t}$. The problem of slow passage through a Hopf bifurcation is ultimately a singular perturbation problem with tiny parameter ϵ, the ramp speed. Baer and Gaekel showed that for a given parameter ramp, p_{crit} can be obtained from the WKB method familiar in physics.

Such work dealt with models that have no spatial structure, for example, the Fitzhugh-Nagumo model of an excitable cell. This model features a parameter I, meant to represent injected current; the system has a Hopf bifurcation with respect to I, and Baer and Gaekel investigated its response to a slow current ramp $I = I_0 + f(\epsilon t)$. In the present work, we focus on two spatially extended systems from neuroscience. The first is a reaction-diffusion model of a passive cable studded with active spines obeying Fitzhugh-Nagumo dynamics. By passive cable we mean that the cable itself is not excitable, but provides a medium through which the spines, which are excitable, communicate. This system models a passive dendrite covered in dendritic spines. The second system is a reaction-diffusion model of an active Fitzhugh-Nagumo cable. By active cable we mean that the cable itself is excitable; this models a neuron's axon, which has embedded in its membrane ion channels which enable it to generate action potentials. For both of these systems, we apply boundary conditions which describe a situation in which a slow current ramp $I = I_0 + f(\epsilon t)$ is injected into one end of the cable, while the other end is sealed to current. Both linear and nonlinear current ramps were investigated.

It is found that the WKB method provides not only the value I_{crit} which a slow current ramp must attain for sustained firing of action potentials to commence, but also the location along the cable at which this instability first shows itself. Furthermore, as I_{crit} varies, so does this location, in a regular way. Hence, by manipulating the current ramp, we can choose I_{crit}, and with it the location along the cable where the approach to sustained oscillations is first apparent. In addition, we explain why the location at which instability first shows itself varies with I_{crit} as it does. We do this by recognizing that the active cable is actually a limiting case of the spiny passive cable, in which we let the stem resistances approach zero while increasing the number of spines. Dendritic spines have bulbous heads and cylindrical stems connecting them to the dendrite; the degree to which they are electrically coupled to the dendrite is determined by their stem resistances.

All WKB predictions of I_{crit} and the location at which instability first shows itself were tested against finite difference solutions of the system. Roundoff error is a serious issue when solving systems with slow paramaeter ramps, and we address this.

References

[1] S. M. Baer and E. M. Gaekel, Physical Review E. **78**, 036205 (2008).

[2] A.I. Neishtadt, Differential Equations. **23**, 1385 (1987).

[3] S. M. Baer, T. Erneux and J. Rinzel, SIAM J. Appl. Math. **49**, 55 (1989).