A Diophantine Equation

The purpose of this activity is to promote and enhance use and understanding of selected Standards for Mathematical Practice, such as making sense of problems and perseverance in solving them, reasoning abstractly and quantitatively, and using appropriate tools strategically.

Concepts used to address the problem:

- Arithmetic
- Polynomial division
- Factoring

Problem:

A point (*x*, y) is called integral if both *x* and *y* are integers.

How many points on the graph of 1/*x* +1/*y* =1/4 are integral points?

Solving the Problem

One can start the discussion by having the students make sense of the problem. That is, students can discuss the following questions:

- What is an integral point (*x*, *y*)?
- What kind of numbers are 1/*x* and 1/*y*?
- Are *x* and *y* integers?
- Can we solve using arithmetic or algebra?
- What values are not permitted for *x* and *y*?

Arithmetic Approach

The problem provides ample opportunities for all students to participate. Students who are reluctant to raise their voice may notice that 1/8 + 1/8 = 1/4, and here we have a solution, *x* = 8 and *y* = 8, meaning that (8, 8) is a point on the graph. Another student may say 1/2 -1/4 = 1/4, producing the solution *x* = 2 and *y* = -4, or the point (2, -4). One student may notice at this moment that (-4, 2) is also a point on the graph. Some student may remember equivalent fraction and rewrite 1/4 = 3/12 = 2/12 + 1/12 = 1/6 + 1/12, to find the point (6, 12). Students may use a different equivalent fraction, for instance, 1/4 = 5/20 = 4/20 + 1/20 = 1/5 + 1/20, to find the point (5, 20).

Although the discussion begins by recognizing the sum and difference of two fractions, it proceeded to a connection to equivalent fractions as an effective tool to find two more solutions. This is an opportunity to talk about "finding all solutions" as a motivation to bring in other approaches to ensure we produce all internal points on the graph.

Polynomial Approach

First, solving for y in the original equation, one can restate the problem as fining the integer solutions to $y = 4x/(x-4)$. Using polynomial division, we can determine the quotient and remainder of $4x$ when divided by $x-4$ (Figure 1).

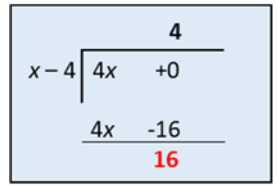

Figure 1.

So, $y = 4x/(x-4) = 4 + 16/(x-4)$. This method provides an opportunity to discuss the usefulness of equivalent expressions. The expression $4 + 16/(x-4)$ offers an alternate approach to solving the problem. Since y must be an integer—and it is expressed as an integer, namely 4, plus a quotient—the quotient itself must be an integer. This occurs when $(x-4)$ is a divisor of 16. As one can notice, this approach allows the teacher to highlight closure properties. From this reasoning, the possibilities for $(x-4)$ are ± 1, ± 2, ± 4, ± 8, and ± 16, leading one to conclude that there are at most ten solutions. The case $x-4 = \pm 4$ gives $x-4 = 4$ or $x = 8$, and $x-4 = -4$ implies $x = 0$. However, based on the conditions of the original problem, $x \neq 0$. Hence, only $x = 8$ give a solution. Substituting $x = 8$ in our last equation, we obtain $y = 4 + 16/((8)-4) = 4 + 16/(4) = 8$. We have found the solution (8, 8).

Proceeding in a similar fashion, we determine all nine solutions: (5,20), (3,-12), (6, 12), (2, -4), (8, 8), (12, 6), (-4, 2), (20, 5), and (-12, 3).

Factoring Approach

Factoring algebraic expressions offers a similar approach to the one we just discussed but without using long division (Figure 2). Again, with the assumption that x and y are both integers, both $(x-4)$ and $4-y)$ are factors (with opposite signs) of -16. Examining all possibilities for these factors, students can determine all nine integer solutions as before. At this moment, the teacher may revisit and stress the connections between factoring and division.

Consider the original equation:

$$1/x + 1/y = 1/4$$

Multiply both sides by 4xy, to eliminate denominators:

$$4y + 4x = xy$$

Subtract xy on both sides:

$$4y + 4x - xy = 0$$

Add -16 to both sides:

$$-16 + 4y + 4x - xy = -16$$

Factor by grouping:

$$-4(4-y) + x(4-y) = -16$$

Factor out the common term:

$$(x-4)(4-y) = -16$$

Figure 2.

Adapted by Greg Wiggins and Suzanne Lenhart from Ichinose and Martinez-Cruz. 2018. Problem Solving + Problem Posing = Mathematical Practices. Mathematics Teacher 111: 505-511.

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