Tutorial: Stochastic Modeling in Biology
Applications of Continuous-Time Markov Chains and Comparison of Discrete-Time and Continuous-Time Processes

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OUTLINE

Part IIII: Continuous-Time Markov Chains - CTMC

- **Summary of Notation, Gillespie Algorithm**

- **Applications:**

(1) Simple Birth and Death Process [MATLAB PROGRAM]

(2) SIR Epidemic Process [MATLAB PROGRAM]

(3) Competition Process

(4) Predation Process

Part IV: Comparison of Stochastic Processes

- **Brief Introduction to SDEs**

- **Applications:**

(1) Simple Birth and Death Process, DTMC, CTMC, SDE

Part III:
Continuous-Time Markov Chains - CTMC
Notation and Terminology

Discrete random variable: $X(t)$, $t \in [0, \infty)$, with values in

$$\{0, 1, 2, \dots N\} \text{ or } \{0, 1, 2, \dots\}$$

Markov property:

Any sequence of real numbers $0 \le t_0 < t_1 < \cdots < t_n < t_{n+1}$,

$$\begin{aligned}
\text{Prob}\{X(t_{n+1}) &= i_{n+1} | X(t_0) = i_0, X(t_1) = i_1, \dots, X(t_n) = i_n\} \\
&= \text{Prob}\{X(t_{n+1}) = i_{n+1} | X(t_n) = i_n\}.
\end{aligned}$$

Probability mass function of $X(t)$: $\{p_i(t)\}_{i=0}^{\infty}$, where

$$p_i(t) = \text{Prob}\{X(t) = i\}.$$

Transition Matrix for the CTMC has Properties similar to DTMC.

Transition probabilities:

$$p_{ji}(t,s) = \mathrm{Prob}\{X(t) = j | X(s) = i\}, \quad s < t$$

Stationary or Homogeneous Transition Probabilities:

$$p_{ji}(t,s) \equiv p_{ji}(t-s)$$

Generally, the transition matrix is a stochastic matrix,

$$\boxed{\sum_{j=0}^{\infty} p_{ji}(t) = 1}$$

unless the process is explosive (blow-up in finite time).

Forward and Backward Kolmogorov Differential Equations.

Generator matrix

$$Q = P'(0)$$

Forward Kolmogorov differential equations expressed in matrix form:

$$\boxed{\frac{dP(t)}{dt} = QP(t),}$$

Backward Kolmogorov differential equations expressed in matrix form:

$$\boxed{\frac{dP(t)}{dt} = P(t)Q,}$$

where $P(t) = (p_{ji}(t))$ is the matrix of transition probabilities and $Q = (q_{ji})$ is the generator matrix.

The Poisson Process

Assumptions in the Poisson process $\{X(t), t \in [0,\infty)\}$:

(1) $X(0) = 0.$

(2) Infinitesimal probabilities:

$$p_{ji}(\Delta t) = \text{Prob}\{X(t+\Delta t) = j | X(t) = i\}$$

$$= \begin{cases} \lambda \Delta t + o(\Delta t), & j = i+1 \\ 1 - \lambda \Delta t + o(\Delta t), & j = i \\ o(\Delta t), & j \geq i+2 \\ 0, & j < i. \end{cases}$$

The probabilities depend only on the length of time Δt.

The Transition Matrix and Generator Matrix for the Poisson Process.

$$P(\Delta t) = \begin{pmatrix} 1-\lambda\Delta t & 0 & 0 & \cdots \\ \lambda\Delta t & 1-\lambda\Delta t & 0 & \cdots \\ 0 & \lambda\Delta t & 1-\lambda\Delta t & \cdots \\ 0 & 0 & \lambda\Delta t & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix} + o(\Delta t).$$

Note column sums of the matrix are one.

$$Q = P'(0) = \begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ \lambda & -\lambda & 0 & \cdots \\ 0 & \lambda & -\lambda & \cdots \\ 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Column sums of Q are zero.

Diagonal elements are nonpositive

Off-diagonal elements are nonnegative.

Poisson Probabilities

Because $X(0) = 0$, it follows that $p_{i0}(t) = p_i(t)$, $dp/dt = Qp$

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t), \quad p_0(0) = 1$$

$$\frac{dp_i(t)}{dt} = -\lambda p_i(t) + \lambda p_{i-1}(t), \quad p_i(0) = 0, \; i \geq 1.$$

The system can be solved sequentially

$$p_0(t) = e^{-\lambda t}, \quad p_1(t) = \lambda t e^{-\lambda t}, \quad p_2(t) = (\lambda t)^2 \frac{e^{-\lambda t}}{2!}.$$

Poisson probability distribution with parameter λt

$$\boxed{p_i(t) = (\lambda t)^i \frac{e^{-\lambda t}}{i!}, \quad i = 0, 1, 2, \ldots.}$$

with mean and variance

$$m(t) = \lambda t = \sigma^2(t).$$

Waiting Times Between Jumps

The distinction between discrete versus continuous time Markov chains is that in DTMC there is a "jump" to a new state at times $1, 2, \ldots,$ but in CTMC the "jump" to a new state may occur at any time $t \geq 0$. The collection of random variables $\{W_i\}$ denote the jump times or waiting times and the times $T_i = W_{i+1} - W_i$ are referred to as the interevent times.

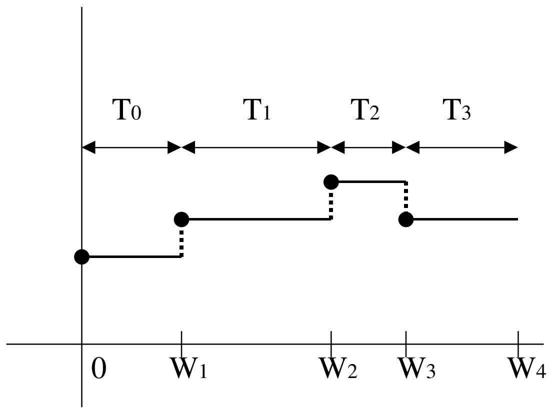


Figure 1: **One sample path of a CTMC, illustrating waiting times and interevent times. The process is continuous from the right.**

An Example of an Explosive Process

If the waiting times approach a positive constant, $W = \sup\{W_i\}$, while the values of the states approach infinity,

$$\boxed{\lim_{i\to\infty} X(W_i) = \infty,}$$

then the process is explosive. We will assume the process is nonexplosive.

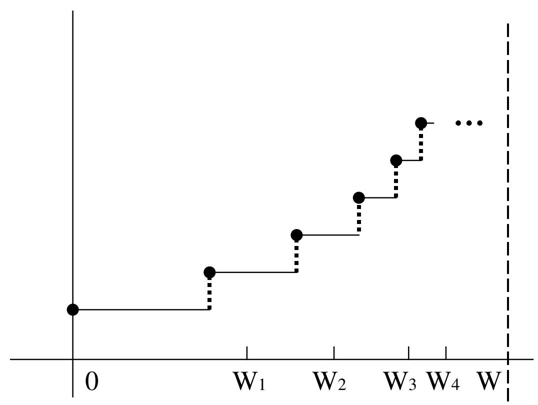


Figure 2: One sample path of a CTMC that is explosive.

Embedded Markov Chain

Waiting times: $\{W_n\}_{n=0}^{\infty}$
Interevent times: $\{T_n\}_{n=0}^{\infty}$, $T_n = W_{n+1} - W_n$
Define a new random variable for state of the CTMC at the nth jump:

$$Y_n = X(W_n), \quad n = 0, 1, 2, \ldots .$$

Embedded MC: $\{Y_n\}_{n=0}^{\infty}$.

The embedded Markov chain is a DTMC, useful for classifying states as transient or recurrent in the associated CTMC.
Example: Transition matrix for embedded MC of Poisson process (all states are transient):

$$T = \begin{pmatrix} 0 & 0 & 0 & \vdots \\ 1 & 0 & 0 & \vdots \\ 0 & 1 & 0 & \vdots \\ 0 & 0 & 1 & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Null Recurrence and Positive Recurrence

Unfortunately, the concepts of null recurrence and positive recurrence for a CTMC cannot be defined in terms of the embedded Markov chain. Positive recurrence depends on the waiting times $\{W_i\}$ so that the embedded Markov chain alone is not sufficient to define positive recurrence.

In an irreducible, recurrent CTMC, let the mean recurrence time for state i be μ_{ii}.

If $\mu_{ii} < \infty$, then the CTMC is positive recurrent.

If $\mu_{ii} = \infty$, then the CTMC is null recurrent.

The Interevent Time for a CTMC Model has an Exponential Distribution.

The exponential waiting time between events characterizes CTMC because the exponential distribution has a memoryless property. The interevent time is $T_i = W_{i+1} - W_i$, where W_i is the time of the ith jump. The event may be a birth, death, immigration, or any other event that changes the value of the state variable. The interevent time $T_i \in [0, \infty)$ is a continuous random variable.

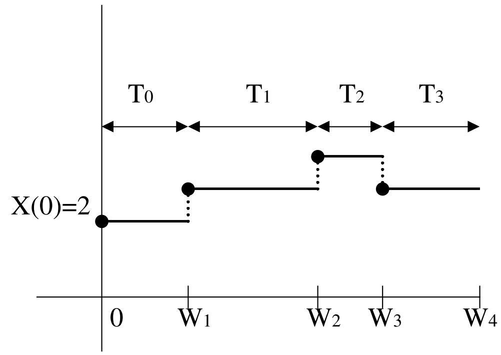


Figure 3: **A sample path or single realization $X(t)$ of a CTMC, $t \in [0, \infty)$ illustrating the jump times $\{W_i\}$ and the interevent times $\{T_i\}$, $X(0) = 2$, $X(W_1) = 3$, $X(W_2) = 4$, $X(W_3) = 3$.**

Interevent Time Theorem.

Theorem 1. Let $\{X(t), t \geq 0\}$, be a CTMC such that

$$\sum_{j=0, j\neq n}^{\infty} p_{jn}(\Delta t) = \alpha(n)\Delta t + o(\Delta t)$$

and

$$p_{nn}(\Delta t) = 1 - \alpha(n)\Delta t + o(\Delta t)$$

for Δt sufficiently small. Then the interevent time, $T_i = W_{i+1} - W_i$ given $X(W_i) = n$, is an exponential random variable with parameter $\alpha(n)$. Cumulative distribution function for T_i is

$$\boxed{F_i(t) = 1 - e^{-\alpha(n)t}.}$$

The mean and variance of T_i are

$$E(T_i) = \frac{1}{\alpha(n)} \quad \text{and} \quad \mathrm{Var}(T_i) = \frac{1}{[\alpha(n)]^2}.$$

Stochastic Realizations

Theorem 2. Let U be a uniform random variable defined on $[0,1]$ and T be a continuous random variable defined on $[0,\infty)$ with cumulative distribution

$$F(t) = \text{Prob}\{T \le t\}.$$

Then $T = F^{-1}(U)$.

Proof. We want to show that $\text{Prob}\{F^{-1}(U) \le t\} = F(t)$. The function F is strictly increasing, so that F^{-1} exists. In addition, for $t \in [0,\infty)$,

$$\begin{aligned} \text{Prob}\{F^{-1}(U) \le t\} &= \text{Prob}\{F(F^{-1}(U)) \le F(t)\} \\ &= \text{Prob}\{U \le F(t)\}. \end{aligned}$$

Because U is a uniform random variable, $\text{Prob}\{U \le y\} = y$ for $y \in [0,1]$. Thus, $\text{Prob}\{U \le F(t)\} = F(t)$. $\square$

In Biochemical Reactions the Preceding Formula is referred to as the Gillespie Algorithm or Stochastic Simulation Algorithm (SSA)

Cumulative Distribution: $F(t) = 1 - e^{-\alpha(n)t}$

Interevent time T:

$$T = F^{-1}(U) = -\frac{\ln(1-U)}{\alpha(n)}$$

$$\boxed{T = -\frac{\ln(U)}{\alpha(n)}}$$

T is small if $\alpha(n)$ is large!

Applications of CTMC

(1) Simple Birth and Death Process

(2) SIR Epidemic Process

(3) Competition Process

(4) Predation Process

(1) Simple Birth and Death Process.

An event can be a birth or a death, $i \to i+1$ or $i \to i-1$. Transition probabilities:

$$\begin{aligned} p_{ji}(\Delta t) &= \text{Prob}\{X(t+\Delta t) = j | X(t) = i\} \\ &= \begin{cases} di\Delta t + o(\Delta t), & j = i-1 \\ bi\Delta t + o(\Delta t), & j = i+1 \\ 1-(b+d)i\Delta t + o(\Delta t), & j = i \\ o(\Delta t), & j \neq i-1,\ i,\ i+1. \end{cases} \end{aligned}$$

Given $X(W_i) = n$, $\alpha(n) = (b+d)n$. Interevent time T_i is

$$\boxed{T_i = -\frac{\ln(U)}{(b+d)n.}}$$

T_i is small if n is large!

(1) Simple Birth and Death Process.

Two events: a birth occurs with probability

$$\frac{b}{b+d}$$

and a death with probability

$$\frac{d}{b+d}.$$

The corresponding ODE is

$$\frac{dn}{dt} = (b-d)n, \quad n(0) = N$$

with solution

$$n(t) = Ne^{(b-d)t}.$$

Three Sample Paths

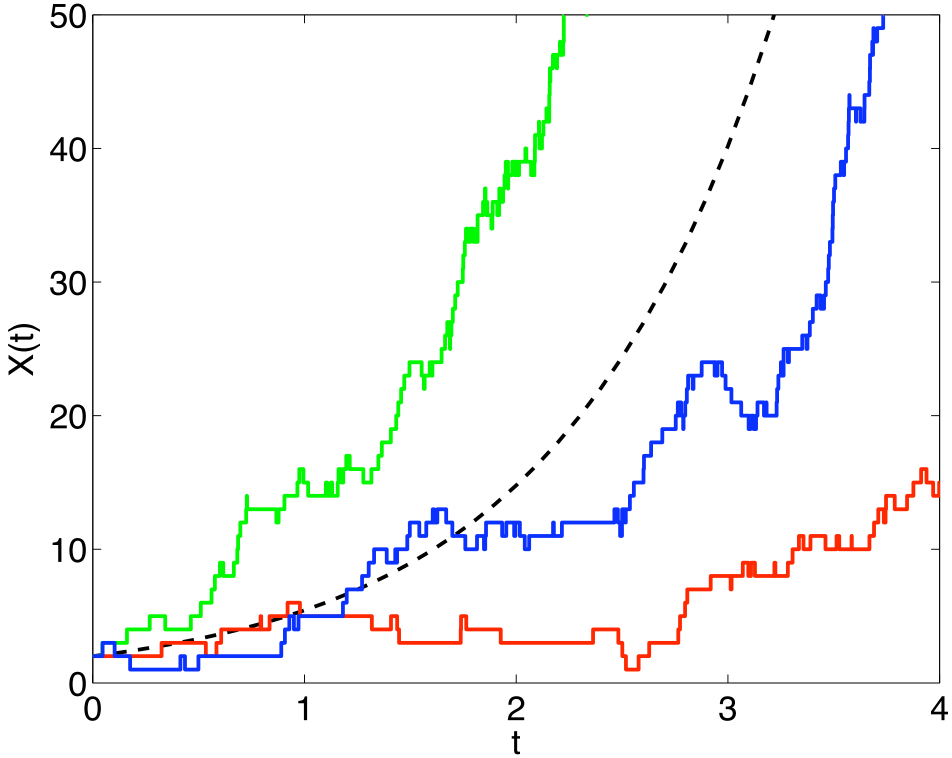


Figure 4: Three sample paths of the simple birth and death process with $X(0) = 2$, $b = 2$, and $d = 1$.

Generator Matrix and Transition Matrix for the Embedded MC

Generator matrix:

$$Q = \begin{pmatrix} 0 & d & 0 & 0 & \cdots \\ 0 & -b-d & 2d & 0 & \cdots \\ 0 & b & -2(b+d) & 3d & \cdots \\ 0 & 0 & 2b & -3(b+d) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Transition matrix for the Embedded MC:

$$T = \begin{pmatrix} 1 & d/(b+d) & 0 & 0 & \cdots \\ 0 & 0 & d/(b+d) & 0 & \cdots \\ 0 & b/(b+d) & 0 & d/(b+d) & \cdots \\ 0 & 0 & b/(b+d) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The embedded Markov chain shows that zero is an absorbing state and the remaining states are transient.

PGF of the Simple Birth and Death Process

An explicit solution to the transition matrix $P(t)$ is not possible but we can determine the moments of this distribution by using a technique known as the generating function technique.

Generating Function Technique: Let the pgf be denoted

$$\mathcal{P}(z,t) = \sum_{i=0}^{\infty} p_i(t) z^i.$$

Multiply the differential equation $dp/dt = Qp$ by z^i and sum:

$$\sum_{i=0}^{\infty} z^i \frac{dp_i(t)}{dt} = \sum_{i=0}^{\infty} z^i \left[b(i-1)p_{i-1}(t) + d(i+1)p_{i+1}(t) - (b+d)ip_i(t) \right].$$

$i = 1, 2, \ldots$ $p_i(0) = \delta_{iN}$. Interchange differentiation and integration:

$$\boxed{\frac{\partial \mathcal{P}}{\partial t} = [d(1-z) + bz(z-1)] \frac{\partial \mathcal{P}}{\partial z}, \quad \mathcal{P}(z,0) = z^N.}$$

Application of the Method of Characteristics

$$\mathcal{P}(z,t) = \begin{cases} \left(\dfrac{e^{t(d-b)}(bz-d) - d(z-1)}{e^{t(d-b)}(bz-d) - b(z-1)}\right)^N, & \text{if } b \neq d \\ \left(\dfrac{1-(bt-1)(z-1)}{1-bt(z-1)}\right)^N, & \text{if } b = d. \end{cases}$$

$p_0(t) = \mathcal{P}(0,t)$:

$$p_0(t) = \begin{cases} \left(\dfrac{d - de^{(d-b)t}}{b - de^{(d-b)t}}\right)^N, & \text{if } b \neq d \\ \left(\dfrac{bt}{1+bt}\right)^N, & \text{if } b = d. \end{cases}$$

Mean and Variance:

$$\boxed{m(t) = Ne^{(b-d)t} \text{ and } \sigma^2(t) = N\frac{(b+d)}{(b-d)}e^{(b-d)t}(e^{(b-d)t}-1),\ b \neq d}$$

$$\boxed{m(t) = N \text{ and } \sigma^2(t) = 2Nbt,\ b = d.}$$

Probability of Extinction in the Simple Birth and Death Process

Probability of extinction, $p_0(t)$, has a simple expression when $t \to \infty$.

Taking the limit,

$$\boxed{p_0(\infty) = \lim_{t\to\infty} p_0(t) = \begin{cases} 1, & \text{if } b \le d \\ \left(\dfrac{d}{b}\right)^N, & \text{if } b > d \end{cases}}$$

See DTMC random walk model with probabilities p and q of moving right or left, respectively.

(2) SIR Epidemic Process

Deterministic Model:

$$\begin{aligned}
\frac{dS}{dt} &= -\beta \frac{S}{N} I \\
\frac{dI}{dt} &= \beta \frac{S}{N} I - \gamma I = I \left(\beta \frac{S}{N} - \gamma \right) \\
\frac{dR}{dt} &= \gamma I,
\end{aligned}$$

$$\boxed{\mathcal{R}_0 = \frac{\beta S(0)/N}{\gamma}}$$

If $\mathcal{R}_0 > 1$, there is an outbreak (increase in number of infectious individuals).
If $\mathcal{R}_0 \leq 1$, there is no outbreak.

In the stochastic model, what is the probability of an outbreak and what is the final size?

Probability of No Outbreak

$$
\begin{aligned}
\text{Prob}\{\Delta S(t) &= i, \Delta I(t) = j | (S(t), I(t))\}) \\
&= \begin{cases}
\dfrac{\beta}{N} S(t) I(t)\, \Delta t + o(\Delta t), & (i,j) = (-1,1) \\
\gamma I(t)\, \Delta t + o(\Delta t), & (i,j) = (0,-1) \\
1 - \left[\dfrac{\beta}{N} S(t) I(t) + \gamma I(t)\right] \Delta t & \\
\quad + o(\Delta t), & (i,j) = (0,0) \\
o(\Delta t), & \text{otherwise.}
\end{cases}
\end{aligned}
$$

Probability of no outbreak (extinction) can be approximated by the birth and death process for large N, $S(0)/N \approx 1$, $\mathcal{R}_0 = \beta/\gamma$, $I(0) = k$:

$$
\boxed{p_0(\infty) \approx \left(\frac{d}{b}\right)^k = \left(\frac{\gamma}{\beta}\right) = \left(\frac{1}{\mathcal{R}_0}\right)^k}
$$

Three Sample Paths of the SIR Stochastic Process, $\mathcal{R}_0 = 2$

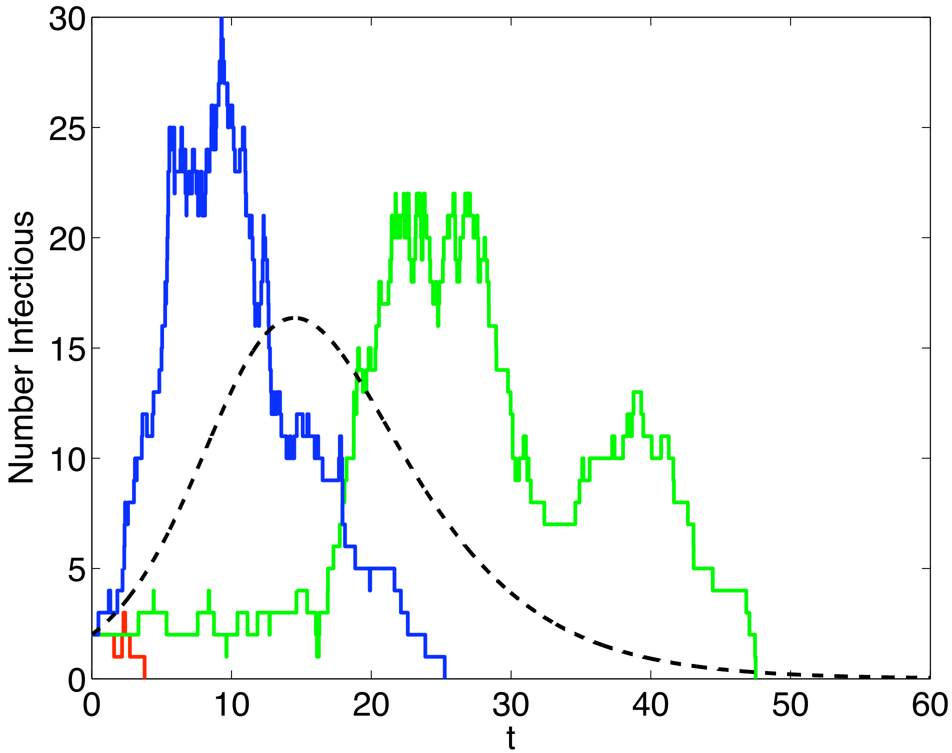


Figure 5: $\beta = 0.5$, $\gamma = 0.25$, $N = 100$, $I(0) = 2$

Final Size in the SIR Model, $\gamma = 1$, $S(0) = N - 1$, and $I(0) = 1$

Deterministic Model

β	N = 20	100	1000
0.5	1.87	1.97	2.00
1	5.74	13.52	44.07
2	16.26	80.02	797.15
5	19.87	99.31	993.03
10	20.00	100.00	999.95

Stochastic Model (Expected Final Size):

β	N = 20	100
0.5	1.76	1.93
1	3.34	6.10
2	8.12	38.34
5	15.66	79.28
10	17.98	89.98

Probability Distribution for the Duration of an Epidemic

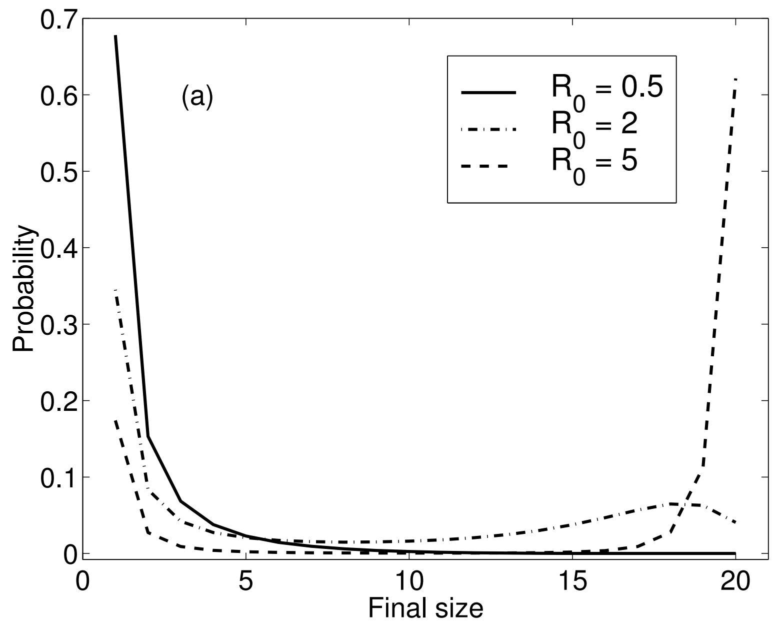


Figure 6: Probability distribution for the final size of a stochastic SIR epidemic model when $I(0) = 1$, $S(0) = 19$, $\gamma = 1$, and $\beta = 0.5$, 2, or 5 ($\mathcal{R}_0 = 0.5$, 2, or 5).

(3) Competition Models

Lotka-Volterra competition, two species compete for the same resource. The deterministic model has the following form:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(a_{10} - a_{11}x_1 - a_{12}x_2) \\ \frac{dx_2}{dt} &= x_2(a_{20} - a_{21}x_1 - a_{22}x_2), \end{aligned}$$

where $x_i(0) > 0$, $a_{ij} > 0$ for $i = 1, 2$ and $j = 0, 1, 2$. There are four different outcomes depending in the parameters and the initial conditions.

Stochastic Competition Process.

Let $X_1(t)$ and $X_2(t)$ be random variables for two competing species, $X_1, X_2 \in \{0, 1, 2, \ldots\}$ and $t \in [0, \infty)$. Let $p_{(i,j)}(t) = \text{Prob}\{X_1(t) = i, X_2(t) = j\}$. Suppose the birth rates are $\lambda_i(X_1, X_2)$ and death rates are $\mu_i(X_1, X_2)$ so that the deterministic model is of the form

$$\frac{dx_i}{dt} = \lambda_i(x_1, x_2) - \mu_i(x_1, x_2), \quad i = 1, 2.$$

One example is

$$\lambda_i(X_1, X_2) = a_{i0} X_i \;\; \text{and} \;\; \mu_i(X_1, X_2) = X_i(a_{i1} X_1 + a_{i2} X_2).$$

$$\text{Prob}\{\Delta X_1(t) = i, \Delta X_2(t) = j | (X_1(t), X_2(t))\}$$
$$= \begin{cases} a_{10} X_1(t)\Delta t + o(\Delta t), & (i,j) = (1,0) \\ a_{20} X_2(t)\Delta t + o(\Delta t), & (i,j) = (0,1) \\ X_1(t)[a_{11}X_1(t) + a_{12}X_2(t)]\Delta t + o(\Delta t), & (i,j) = (-1,0) \\ X_2(t)[a_{21}X_1(t) + a_{22}X_2(t)]\Delta t + o(\Delta t), & (i,j) = (0,-1) \\ 1 - X_1(t)[a_{11}X_1(t) + a_{12}X_2(t)]\Delta t & \\ \quad - X_2(t)[a_{21}X_1(t) + a_{22}X_2(t)]\Delta t + o(\Delta t), & (i,j) = (0,0) \\ o(\Delta t), & \text{otherwise.} \end{cases}$$

Stochastic Competition Process.

It follows from forward Kolmogorov differential equations,

$$\begin{aligned}\frac{dp_{(i,j)}}{dt} &= \lambda_1(i-1,j)p_{(i-1,j)} + \lambda_2(i,j-1)p_{(i,j-1)} \\ &\quad + \mu_1(i+1,j)p_{(i+1,j)} + \mu_2(i,j+1)p_{(i,j+1)} \\ &\quad - [\lambda_1(i,j) + \lambda_2(i,j) + \mu_1(i,j) + \mu_2(i,j)]\, p_{(i,j)}.\end{aligned}$$

Differential equations for the means are

$$\begin{aligned}\frac{dE[X_1(t)]}{dt} &= a_{10}E[X_1(t)] - a_{11}E[(X_1(t))^2] - a_{12}E[X_1(t)X_2(t)] \\ \frac{dE[X_2(t)]}{dt} &= a_{20}E[X_2(t)] - a_{21}E[X_1(t)X_2(t)] - a_{22}E[(X_2(t))^2].\end{aligned}$$

The two differential equations for the means depend on five unknown variables and cannot be solved explicitly. However, the form of these equations is similar to the deterministic differential equations. Specific assumptions (e.g., normality or lognormality) about $E[(X_1(t))^k(X_2(t))^l]$ are required to approximate the higher-order moments of the distribution known moment closure assumptions.

An Example of a Competition Process.

Let $a_{10} = 2$, $a_{20} = 1.5$, $a_{11} = 0.03$, $a_{12} = 0.02$, $a_{21} = 0.01$, and $a_{22} = 0.04$. A stable positive equilibrium exists,

$$(\bar{x}_1, \bar{x}_2) = (50, 25)$$

At $t = 5$, the means and variances are estimated from 1000 sample paths,

$$m_{X_1}(5) = 49.9, \quad m_{X_2}(5) = 23.2,$$

$$\sigma_{X_1}(5) = 9.4, \quad \sigma_{X_2}(5) = 6.8.$$

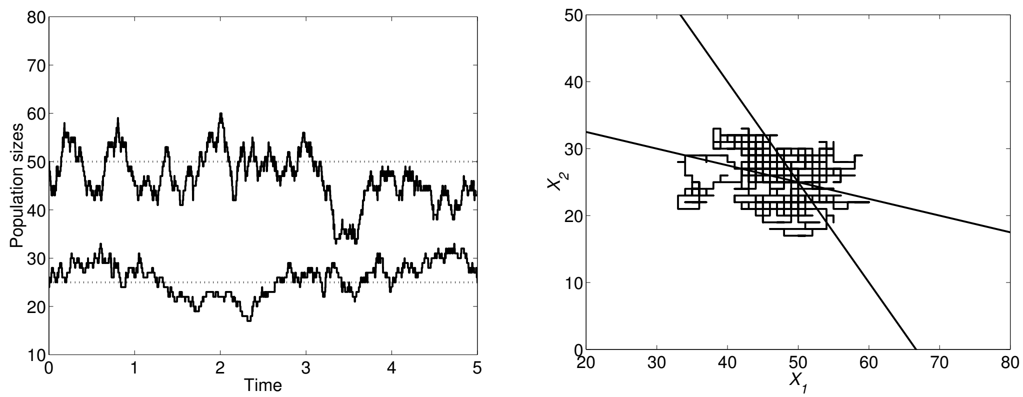


Figure 7: $a_{10} = 2$, $a_{20} = 1.5$, $a_{11} = 0.03$, $a_{12} = 0.02$, $a_{21} = 0.01$, $a_{22} = 0.04$, $X_1(0) = 50$, and $X_2(0) = 25$. **The dotted lines indicate the equilibrium values.**

(4) Predator-Prey Process.

The Lotka-Volterra predator-prey model has the form

$$\begin{aligned}\frac{dx}{dt} &= x(a_{10} - a_{12}y)\\ \frac{dy}{dt} &= y(a_{21}x - a_{20}),\end{aligned}$$

where $a_{ij} > 0$. The equilibrium is neutrally stable:

$$(a_{20}/a_{21}, a_{10}/a_{12})$$

Predator-Prey Process.

Let $X(t)$ and $Y(t)$ denote random variables for the size of the prey and predator populations respectively, in a stochastic Lotka-Volterra model. Assume the transition probabilities satisfy

$$\mathrm{Prob}\{\Delta X(t) = i, \Delta Y(t) = j | (X(t), Y(t))\}$$
$$= \begin{cases} a_{10}X(t)\Delta t + o(\Delta t), & (i,j) = (1,0) \\ a_{21}X(t)Y(t)\Delta t + o(\Delta t), & (i,j) = (0,1) \\ a_{12}X(t)Y(t)\Delta t + o(\Delta t), & (i,j) = (-1,0) \\ a_{20}Y(t)\Delta t + o(\Delta t), & (i,j) = (0,-1) \\ 1 - X(t)[a_{10} + a_{12}Y(t)]\Delta t & \\ \; - Y(t)[a_{20} + a_{21}X(t)]\Delta t + o(\Delta t), & (i,j) = (0,0) \\ o(\Delta t), & \text{otherwise.} \end{cases}$$

An Example of the Lotka-Volterra Predation Process.

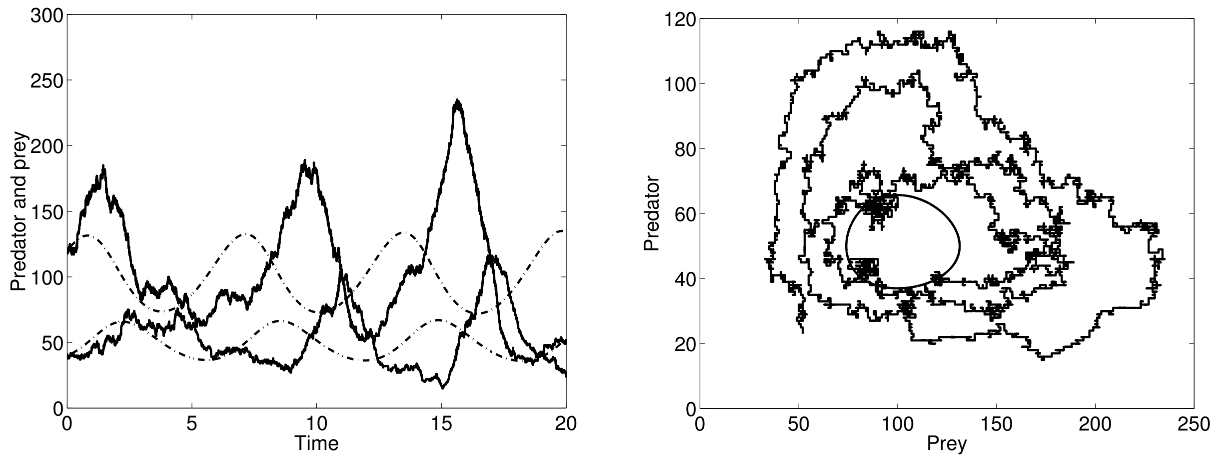


Figure 8: **A sample path of the Lotka-Volterra predator-prey model is compared to the solution of the deterministic model. Solutions are graphed over time and in the phase plane. The parameter values and initial conditions satisfy $a_{10} = 1$, $a_{20} = 1$, $a_{12} = 0.02$, $a_{21} = 0.01$, $X(0) = 120$, and $Y(0) = 40$. Solutions with the smaller amplitude represent the predator. Equilibrium $(100, 50)$.**

Part IV:

Comparison of Stochastic Processes
Introduce Diffusion Process and SDEs

Continuous Random Variable: $X(t)$, $t \in [0, \infty)$ with values in

$$(-\infty, \infty) \quad \text{or} \quad [0, \infty) \quad \text{or} \quad [0, M].$$

Markov property: Given any sequence of times, $t_0 < t_1 < \cdots < t_{n-1} < t_n$,

$$\begin{aligned} \text{Prob}\{X(t_n) \le y | X(t_0) &= x_0, X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1}\} \\ &= \text{Prob}\{X(t_n) \le y | X(t_{n-1}) = x_{n-1}\}. \end{aligned}$$

Probability density function (pdf) of $X(t)$: $p(x,t)$,

$$\text{Prob}\{X(t) \in [a,b]\} = \int_a^b p(x,t)\, dx.$$

Discrete Random Walk and the Diffusion Equation

Consider a random walk on the set $\{0, \pm\Delta x, \pm 2\Delta x, \ldots\}$. Let $p =$ probability of moving to the right and $q =$ probability of moving to the left, $p + q = 1$. Let $\{X(t)\}$ be DTMC for this random walk, where $t \in \{0, \Delta t, 2\Delta t, \ldots\}$, $X(t) \in \{0, \pm\Delta x, \pm 2\Delta x, \ldots\}$, and $p_x(t) = \mathrm{Prob}\{X(t) = x\}$. Define $u(x,t) = p_x(t)$. It follows that

$$u(x, t + \Delta t) = pu(x - \Delta x, t) + qu(x + \Delta x, t).$$

Expanding the right-hand side of the preceding equation using Taylor's formula about the point (x,t),

$$\begin{aligned} u(x, t+\Delta t) &= p\left[u(x,t) + \frac{\partial u(x,t)}{\partial x}(-\Delta x) + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2} + O((\Delta x)^3)\right] \\ &\quad + q\left[u(x,t) + \frac{\partial u(x,t)}{\partial x}\Delta x + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2} + O((\Delta x)^3)\right] \\ &= u(x,t) + (q-p)\frac{\partial u(x,t)}{\partial x}\Delta x + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2} + O((\Delta x)^3). \end{aligned}$$

Subtracting $u(x,t)$ and dividing by Δt,

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = (q-p)\frac{\partial u(x,t)}{\partial x}\frac{\Delta x}{\Delta t} + \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{\Delta t} + O\left(\frac{(\Delta x)^3}{\Delta t}\right).$$

We make the assumptions

$$\boxed{\begin{aligned}\lim_{\Delta t,\Delta x\to 0}\frac{(\Delta x)^3}{\Delta t} &= 0\\ \lim_{\Delta t,\Delta x\to 0}(p-q)\frac{\Delta x}{\Delta t} &= c,\\ \lim_{\Delta t,\Delta x\to 0}\frac{(\Delta x)^2}{\Delta t} &= D.\end{aligned}}$$

Letting Δt and Δx approach zero, then

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + \frac{D}{2}\frac{\partial^2 u}{\partial x^2}, \quad x \in (-\infty, \infty).$$

the **diffusion equation with drift**, where c = drift coefficient D = diffusion coefficient. When $p = 1/2 = q$, symmetric, the limiting stochastic process is known as **Brownian motion**, $c = 0$,

$$\frac{\partial u}{\partial t} = \frac{D}{2}\frac{\partial^2 u}{\partial x^2}, \quad x \in (-\infty, \infty).$$

If $D = 1$, **Standard Brownian motion** or **Wiener Process.**

Wiener Process

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \quad x \in (-\infty, \infty).$$

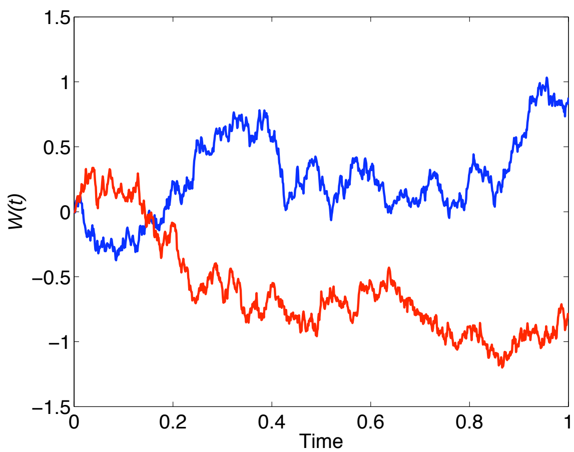
For $u(x,0) = \delta(x)$, the solution is the pdf of the Wiener process $W(t)$:

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad x \in (-\infty, \infty)$$

which is also the pdf of a Normal distribution with mean zero and variance t:

$$W(t) \sim N(0,t).$$

Sample paths of the Wiener process are continuous but nowhere differentiable.



The Forward and Backward Kolmogorov Differential Equations Follow From these Assumptions.

The **backward Kolmogorov differential equation** for a time-homogeneous process is

$$\frac{\partial p(y,x,t)}{\partial t} = a(x)\frac{\partial p(y,x,t)}{\partial x} + \frac{1}{2}b(x)\frac{\partial^2 p(y,x,t)}{\partial x^2}.$$

The **forward Kolmogorov differential equation** for a time-homogeneous process is

$$\frac{\partial p(y,x,t)}{\partial t} = -\frac{\partial\left[a(y)p(y,x,t)\right]}{\partial y} + \frac{1}{2}\frac{\partial^2\left[b(y)p(y,x,t)\right]}{\partial y^2}.$$

An Itô Stochastic Differential Equation (SDE)

The pdf $p(x,t)$ with $p(x,0) = \delta(x - x_0)$ then $p(x,t)$ is a solution of the forward Kolmogorov differential equation:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial \left[a(x,t)p(x,t)\right]}{\partial x} + \frac{1}{2}\frac{\partial^2 \left[b(x,t)p(x,t)\right]}{\partial x^2}.$$

A sample path of the process $\{X(t), t \in [0,\infty)\}$ is a solution of the Itô SDE of the form:

$$\boxed{dX(t) = a(X(t),t)dt + \sqrt{b(X(t),t)}dW(t), \quad X(0) = x_0}$$

where $W(t)$ is a Wiener process.

Some Comparison of the Simple Birth and Death Process

CTMC Transition probabilities:

$$
\begin{aligned}
p_{ji}(\Delta t) &= \text{Prob}\{X(t+\Delta t) = j | X(t) = i\} \\
&= \begin{cases} di\Delta t + o(\Delta t), & j = i-1 \\ bi\Delta t + o(\Delta t), & j = i+1 \\ 1-(b+d)i\Delta t + o(\Delta t), & j = i \\ o(\Delta t), & j \neq i-1,\ i,\ i+1. \end{cases}
\end{aligned}
$$

DTMC and CTMC have the same probability of extinction:

$$
\boxed{p_0(\infty) = \begin{cases} 1, & b \le d \\ \left(\dfrac{d}{b}\right)^k, & b > d \end{cases}}
$$

SDE for a Birth and Death Process:

$$dX(t) = (b-d)X\,dt + \sqrt{(b+d)X(t)}\,dW(t), \;\; X(0) = k.$$

CTMC and SDE have the same mean and variance. The mean is the same as the ODE model ($b \neq d$):

$$\boxed{E(X) = ke^{(b-d)t}, \;\; Var(X) = k\frac{(b+d)}{b-d}e^{(b-d)t}(e^{(b-d)t}-1)}$$

See Lectures by E.Allen, J. Xiong and J. M. Ponciana

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