Synchrony, Phase-Shift Synchrony, and Synchony-Breaking

NIMBioS
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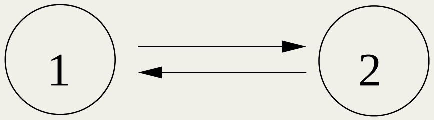
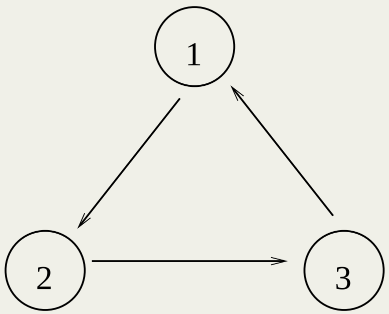
Marty Golubitsky
Mathematical Biosciences Institute
and
Department of Mathematics
Ohio State University

Networks and Coupled Systems (1)



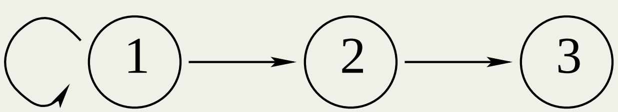
$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2)\\ \dot{x}_2 &= f(x_2, x_1)\end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$

$\Delta = \{x_1 = x_2\}$ is flow-invariant



$$\begin{aligned}\dot{x}_1 &= f(x_1, x_3)\\ \dot{x}_2 &= f(x_2, x_1)\\ \dot{x}_3 &= f(x_3, x_2)\end{aligned} \qquad x_1, x_2, x_3 \in \mathbf{R}^k$$

$\Delta = \{x_1 = x_2 = x_3\}$ is flow-invariant



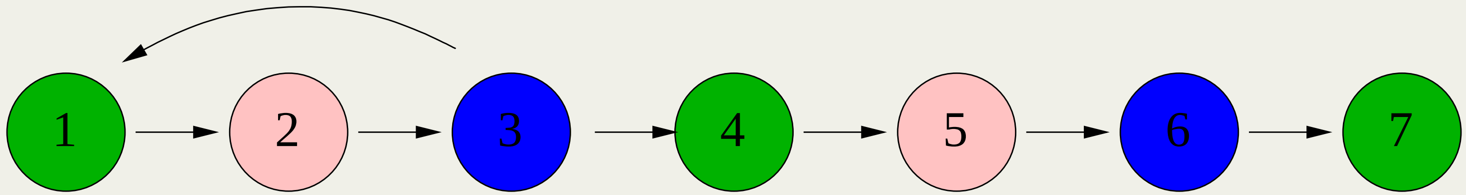
$$\begin{aligned}\dot{x}_1 &= f(x_1, x_1, \lambda)\\ \dot{x}_2 &= f(x_2, x_1, \lambda)\\ \dot{x}_3 &= f(x_3, x_2, \lambda)\end{aligned} \qquad x_1, x_2, x_3 \in \mathbf{R}^k$$

$\Delta = \{x_1 = x_2 = x_3\}$ is flow-invariant

Synchrony Subspaces

- Polydiagonal is subspace $\Delta = \{x_c = x_d \quad \text{for some subset of cells}\}$
- **Synchrony subspace** is flow-invariant polydiagonal
- Synchrony subspace corresponds to solutions with EXACT synchrony between subsets of nodes

Chain with Back Coupling

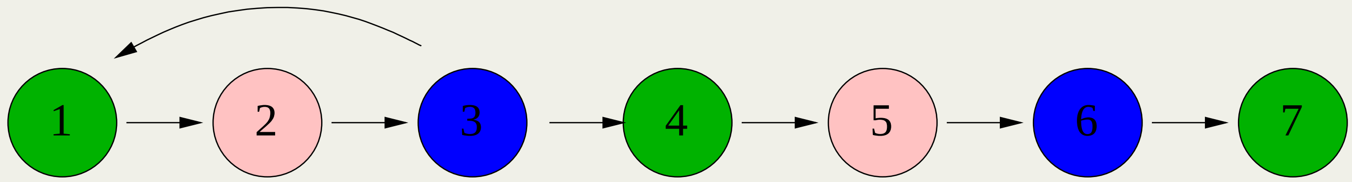


$$
\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, \mathbf{x}_3) & \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, \mathbf{x}_1) & \dot{\mathbf{x}}_3 &= \mathbf{f}(\mathbf{x}_3, \mathbf{x}_2) \\
\dot{\mathbf{x}}_4 &= \mathbf{f}(\mathbf{x}_4, \mathbf{x}_3) & \dot{\mathbf{x}}_5 &= \mathbf{f}(\mathbf{x}_5, \mathbf{x}_4) & \dot{\mathbf{x}}_6 &= \mathbf{f}(\mathbf{x}_6, \mathbf{x}_5) \\
\dot{\mathbf{x}}_7 &= \mathbf{f}(\mathbf{x}_7, \mathbf{x}_6)
\end{aligned}
$$

- $\boldsymbol{\Delta} = \{\mathbf{x}_1 = \mathbf{x}_4 = \mathbf{x}_7;\ \mathbf{x}_2 = \mathbf{x}_5;\ \mathbf{x}_3 = \mathbf{x}_6\}$ is flow-invariant
- $\boldsymbol{\Delta}$ is a synchrony subspace

Balanced Coloring

- Let Δ be a polydiagonal
- Color equivalent cells the same color if cell coord's in Δ are equal
- Coloring is **balanced** if all cells with same color receive equal number of inputs from cells of a given color

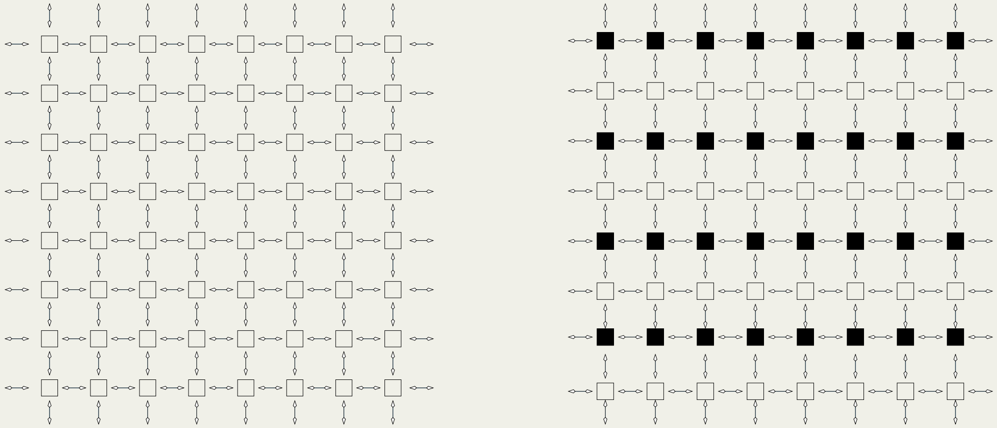


- **Theorem 1**: **synchrony subspace $\Longleftrightarrow$ balanced**

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

2D-Lattice Dynamical Systems

- square lattice with nearest neighbor coupling
- Network architecture is more important than symmetry
- Form two-color balanced relation

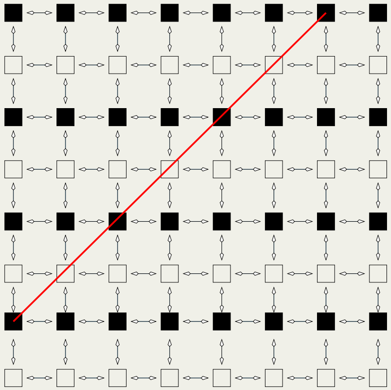
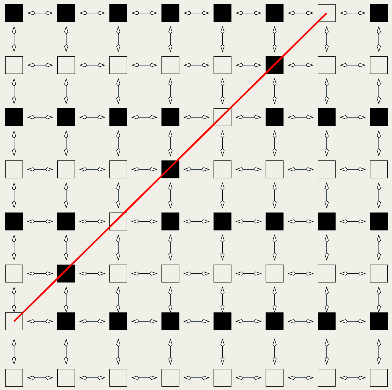


- Each black cell connected to two black and two white
  Each white cell connected to two black and two white

Stewart, G. and Nicol (2004)

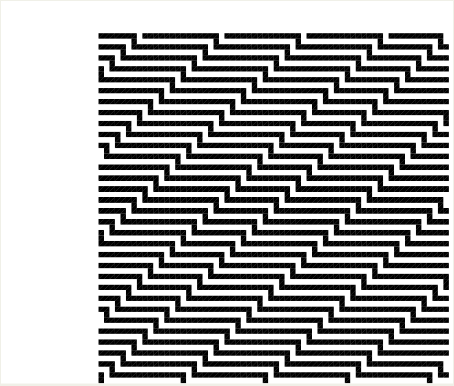
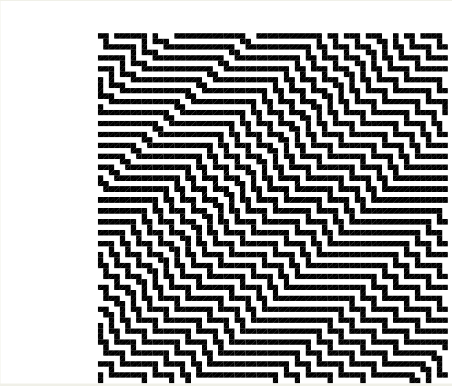
Lattice Dynamical Systems

- On Black/White diagonal interchange black and white

Result is balanced

- Continuum of different synchrony subspaces

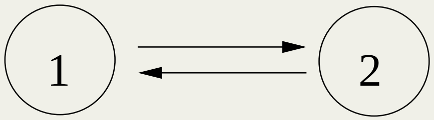



Lattice Dynamical Systems

- Architecture is important
- For square lattice with nearest and next nearest neighbor coupling
  - No infinite families
  - For each k a finite number of balanced k colorings
  - All balanced colorings are doubly-periodic

Antoneli, Dias, G., and Wang (2004)

Phase-Shift Synchrony: Two Identical Cells



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1) \end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$

- Rigid time-periodic solutions exist where cells oscillate in phase

$$x_2(t) = x_1(t)$$

  Not surprising since $x_1 = x_2$ is flow-invariant

- Robust time-periodic solutions exist where cells oscillate a half-period out-of-phase

$$x_2(t) = x_1(t + \frac{T}{2})$$

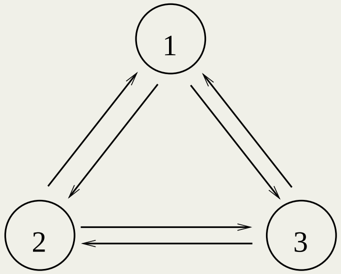
Spatio-Temporal Symmetries

- A symmetry of $\dot{x} = F(x)$ is linear map γ that takes sol'ns to sol'ns
- Let $x(t)$ be a time-periodic solution

$H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\}$ **spatiotemporal symm's**

- $\gamma \in H \Longrightarrow \theta \in \mathbf{S}^1$ such that $\gamma x(t) = x(t + \theta)$
- Example: $H = \mathbf{Z}_2(1\ 2)$; $\theta = 0$ or $\theta = \frac{T}{2}$

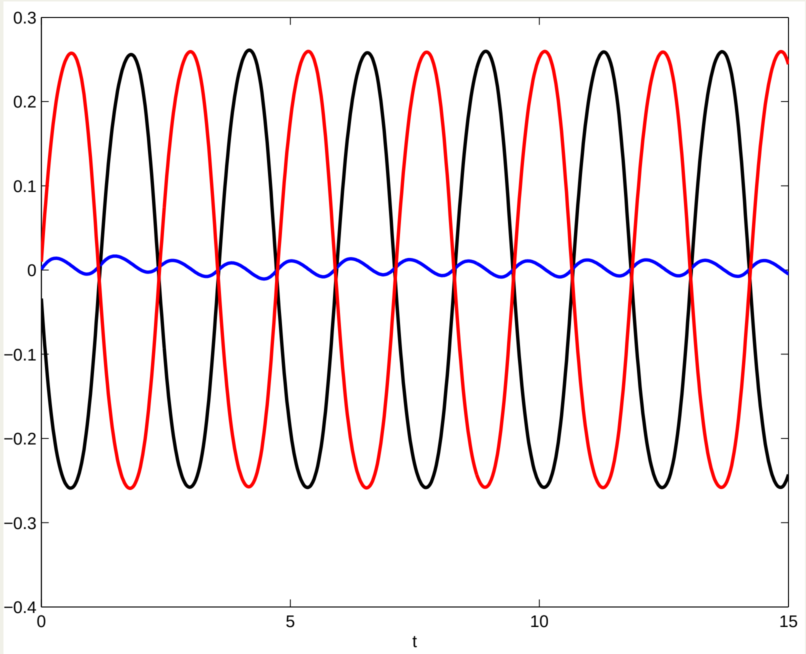
Three-Cell Bidirectional Ring: $\Gamma = \mathbf{S}_3$



$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2, x_3)\\ \dot{x}_2 &= f(x_2, x_3, x_1) \qquad f(x_2, x_1, x_3) = f(x_2, x_3, x_1)\\ \dot{x}_3 &= f(x_3, x_1, x_2)\end{aligned}$$

- Out-of-phase: $H =< (1\ 3)(2) >$

$$x_3(t) = x_1\left(t + \frac{T}{2}\right) \quad \text{and} \quad x_2(t) = x_2\left(t + \frac{T}{2}\right)$$

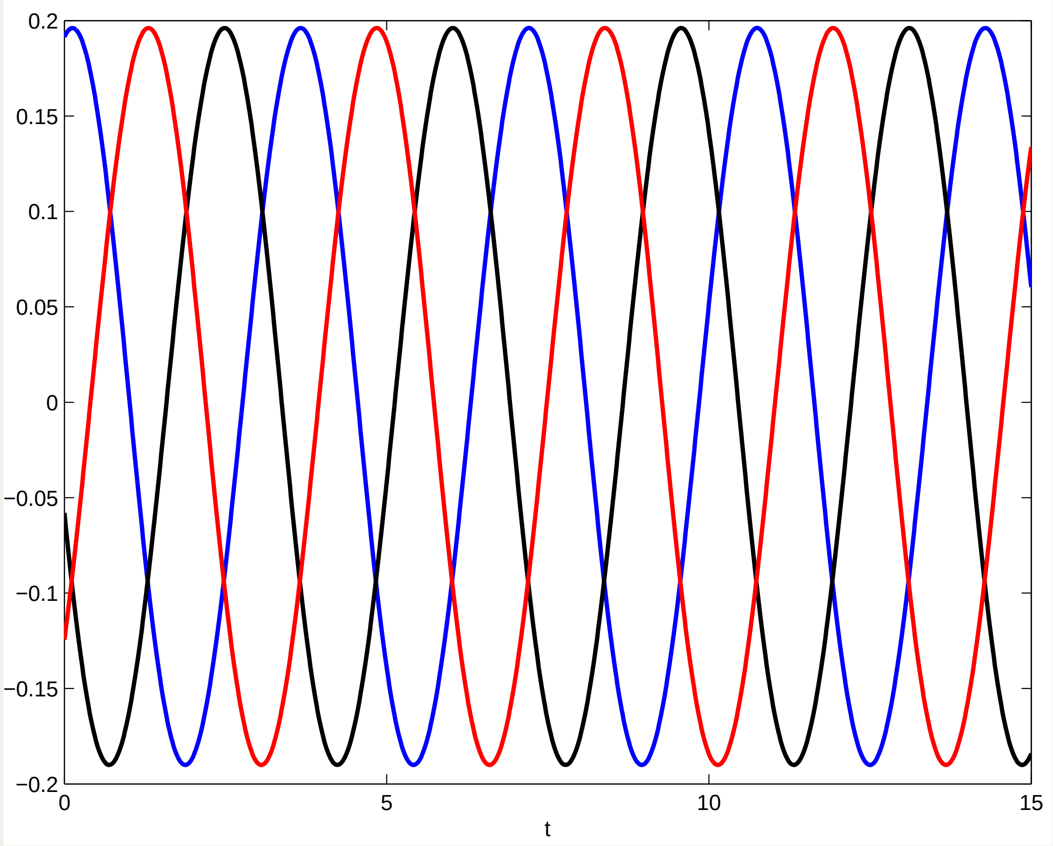


G. and Stewart (1986)

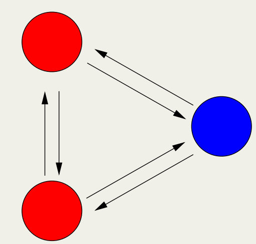
Discrete Rotating Wave

- Out-of-phase: $H =< (1\ \ 3) >$

$x_2(t) = x_1\left(t + \frac{T}{3}\right)$ and $x_3(t) = x_2\left(t + \frac{T}{3}\right) = x_1\left(t + \frac{2T}{3}\right)$



Quotient Networks: Self-Coupling & Multiarrows

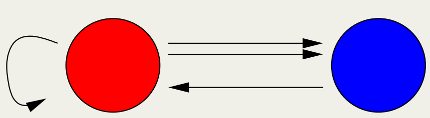


- Balanced two-coloring of bidirectional ring

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1) \\ \dot{\mathbf{x}}_3 &= \mathbf{f}(\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_2) \end{aligned} \qquad \text{where } \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

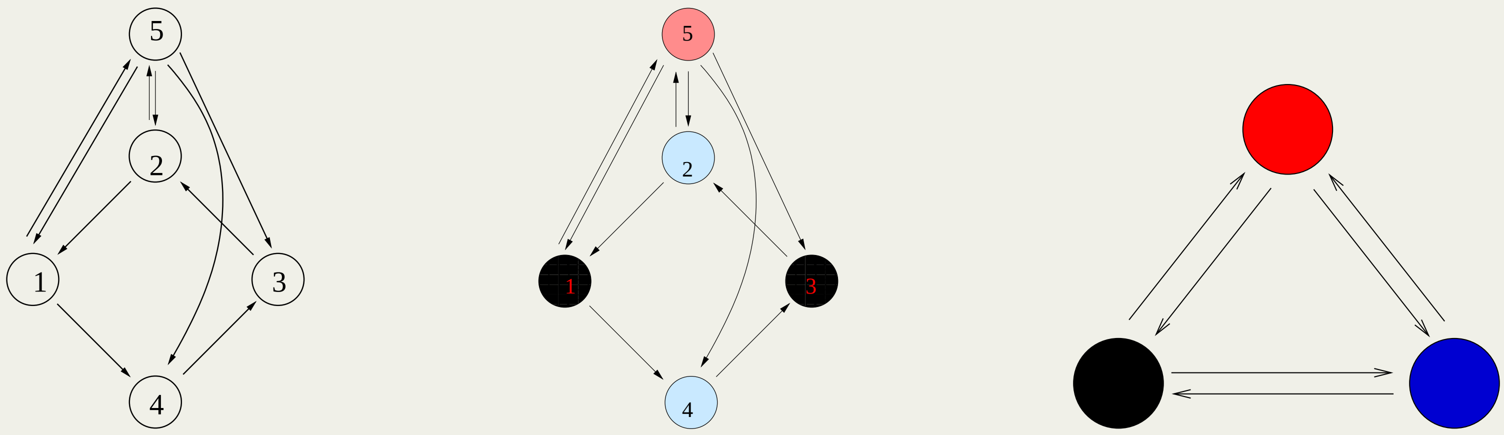
$\Delta = \{x_1 = x_2\}$ is a synchrony subspace

- Quotient network:

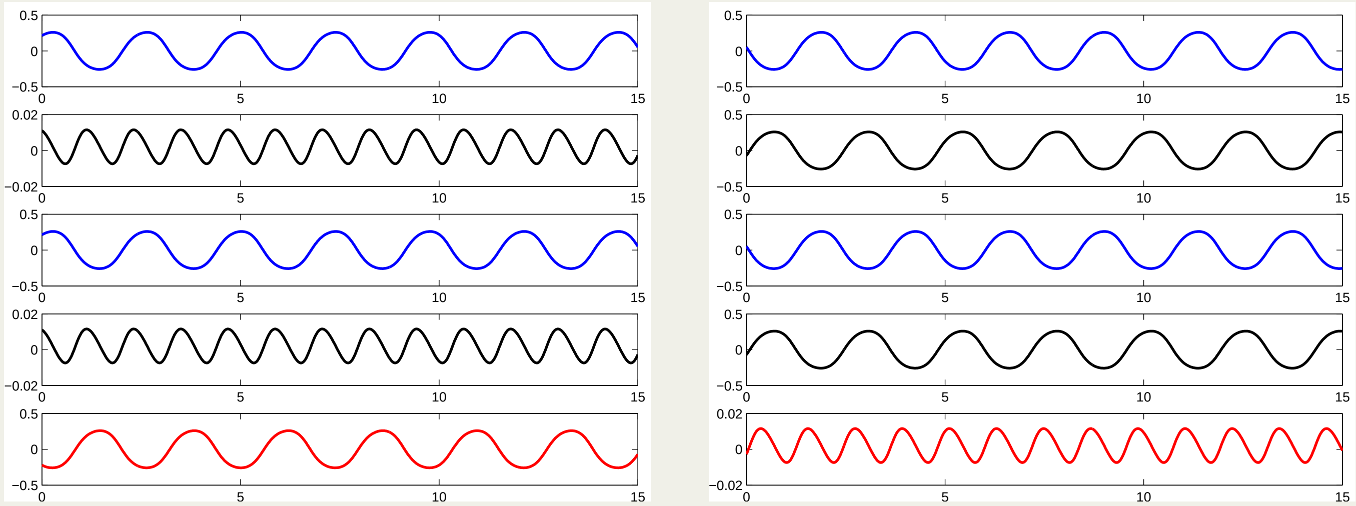


$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_3) \\ \dot{\mathbf{x}}_3 &= \mathbf{f}(\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_1) \end{aligned} \qquad \text{where } \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

Asym Network; Symmetric Quotient



- Quotient is bidirectional 3-cell ring with $\mathbf{D}_3$ symmetry



- **Rigid phase shift; no symmetry**

Pattern of Phase-Shift Synchrony

- Let $\mathcal{G}$ be a network
- **Pattern of phase-shift synchrony** is quotient network $\mathcal{Q}$ and permutation symmetry $\sigma : \mathcal{Q} \to \mathcal{Q}$
- A T-periodic solution $Z(t)$ to a $\mathcal{G}$-admissible system **has** pattern of phase-shift synchrony $\mathcal{Q}$ and $\sigma)$ if
  - $\{Z(t)\} \subset \Delta_{\mathcal{Q}}$; $Y(t)$ is $Z(t)$ viewed in quotient network
  - $\sigma Y(t) = Y\left(t + \frac{T}{m}\right)$ where m is order of σ

Consequences of Pattern of Phase-Shift Synchrony

- $\{Z(t)\} \subset \Delta_{\mathcal{Q}} \Longrightarrow z_c(t) = z_d(t)$ when nodes $c, d \in \mathcal{G}$ have same color

- $\sigma = \sigma_1 \cdots \sigma_s$ product of disjoint cycles of orders $m_1, \ldots, m_s \leq m$

- Renumber nodes in $\mathcal{Q}$ so that $\sigma_1 = (1 \cdots m_1)$. Let $Y(t)$ be $Z(t)$ viewed in $\mathcal{Q}$. Then $\sigma Y(t) = Y(t + \frac{T}{m})$ implies

$$\begin{aligned} y_2(t) &= y_1(t + \tfrac{T}{m}) \\ &\vdots \\ y_{m_1}(t) &= y_{m_1-1}(t + \tfrac{T}{m}) \\ y_1(t) &= y_{m_1}(t + \tfrac{T}{m}) \end{aligned}$$

- So $y_1(t) = y_1(t + \frac{m_1}{m}T)$ and y_1 has period $T_1 = \frac{m_1}{m}T$

- Cycles of different lengths in σ imply multirhythms

Rigid Phase-Shift $\Leftrightarrow$ Pattern of Phase-Shift Synchrony

- $Z(t) = (z_1(t), \ldots, z_N(t))$ is hyperbolic T-periodic solution

- **Phase-shift synchrony** between nodes i, j

$$z_i(t) = z_j(t + \theta T) \text{ where } 0 \leq \theta < 1$$

- Phase-shift synchrony is **rigid** if perturbing system leads to periodic solution with same phase-shift θ

- **Theorem 2:** Assume path-connected network $\mathcal{G}$. Nonzero rigid phase-shift synchrony iff phase-shift forced by some symmetry on a quotient network

Stewart and Parker (2008, 2009); G., Romano and Wang (2010, 2011)

Regular Three Cell Networks

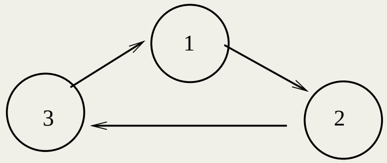
- **Regular network**: one type of node and one type of coupling
- *Valency* $= \nu =$ total number of inputs per cell

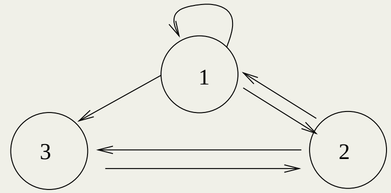
$$a_{i1} + a_{i2} + a_{i3} = \nu \qquad \text{for} \quad j = 1, 2, 3$$

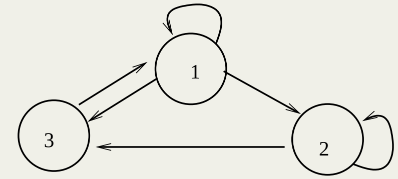
- 34 regular three-cell valency 2 networks

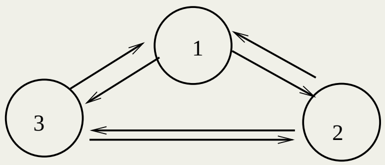
Leite and G. (2006)

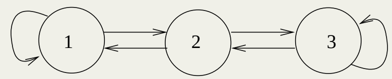
13 Three-Cell Transitive Networks

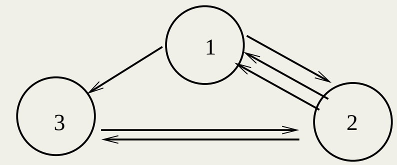
2. 

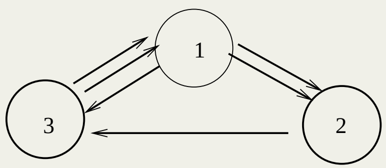
5. 

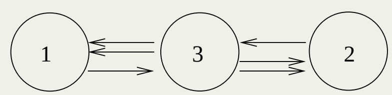
6. 

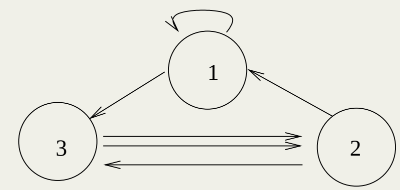
8. 

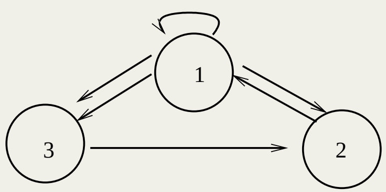
10. 

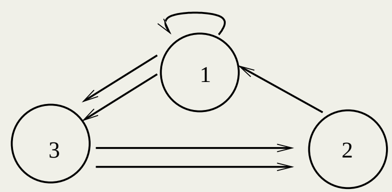
11. 

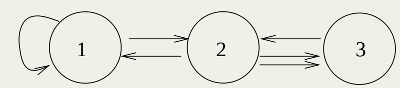
14. 

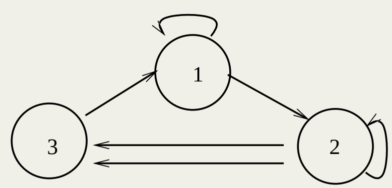
15. 

17. 

18. 

19. 

23. 

24. 

21 Three-Cell Feed-Forward Networks

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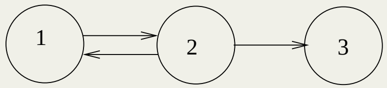

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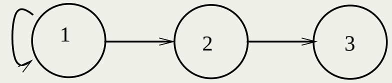

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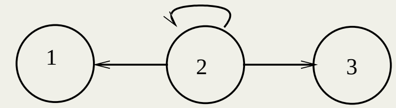

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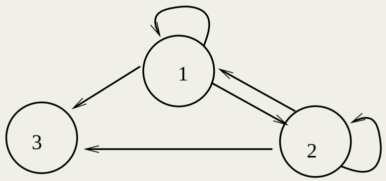

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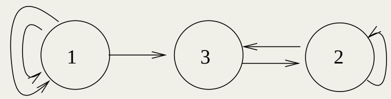

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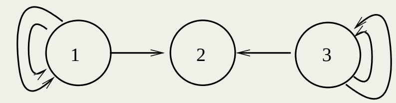

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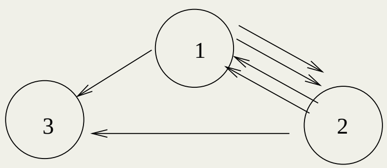

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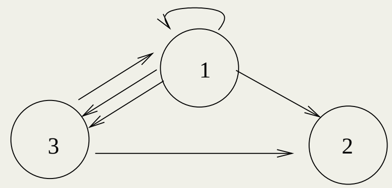

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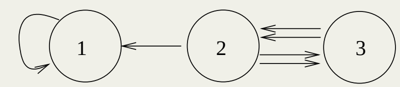

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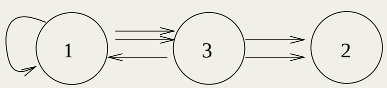

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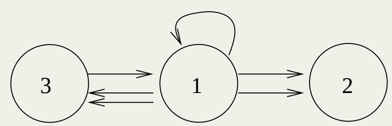

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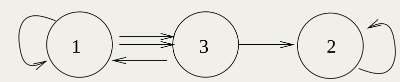

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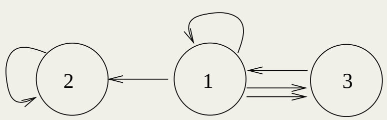

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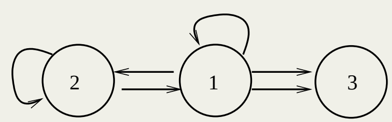

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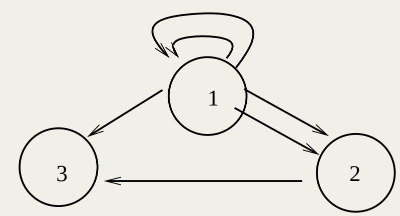

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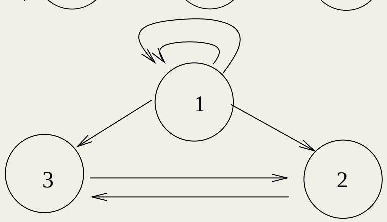

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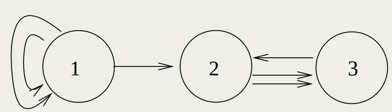

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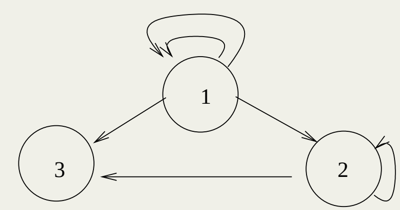

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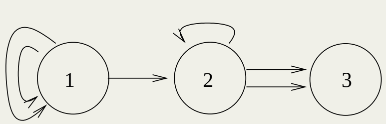

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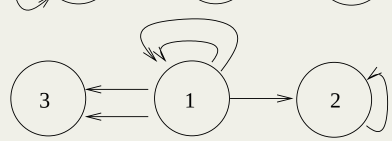

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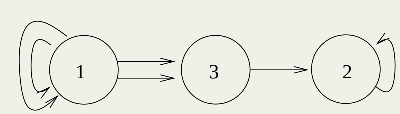
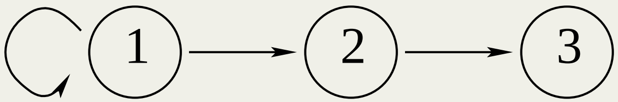


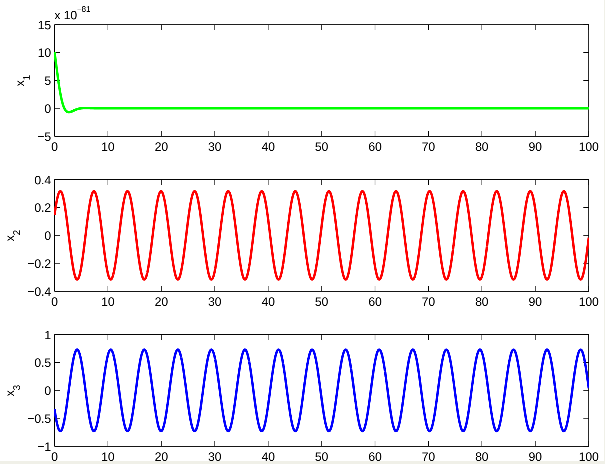
Three-Cell Feed-Forward Network

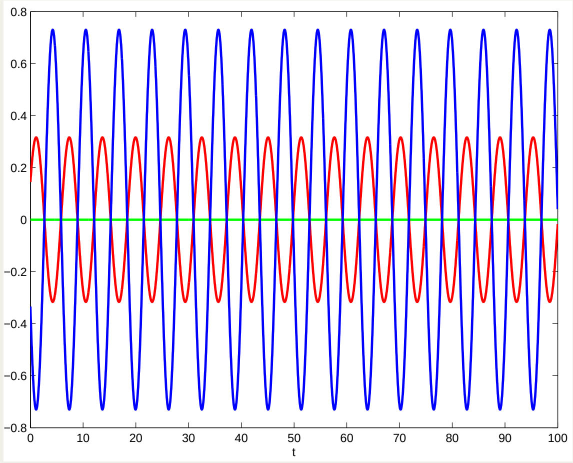
- 

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1, \lambda) \\ \dot{x}_2 &= f(x_2, x_1, \lambda) \\ \dot{x}_3 &= f(x_3, x_2, \lambda) \end{aligned} \qquad J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

- Network supports solution by Hopf bifurcation where
  $x_1(t)$ equilibrium $\qquad x_2(t), x_3(t)$ time periodic

- $x_2(t) \approx \lambda^{1/2} \qquad x_3(t) \approx \lambda^{1/6}$

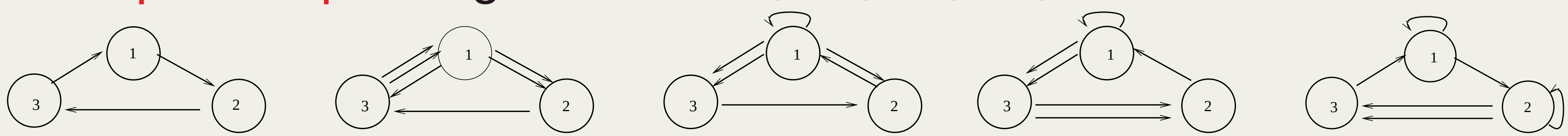




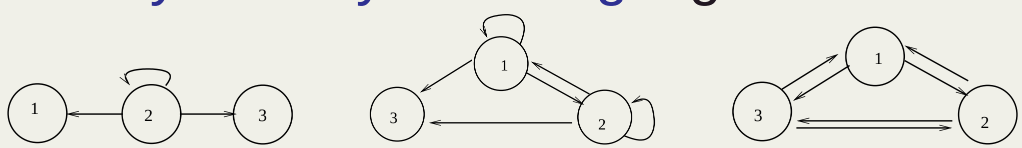
G., Nicol, and Stewart (2004); Elmhirst and G. (2005)

Eigenspace Types of Jacobians

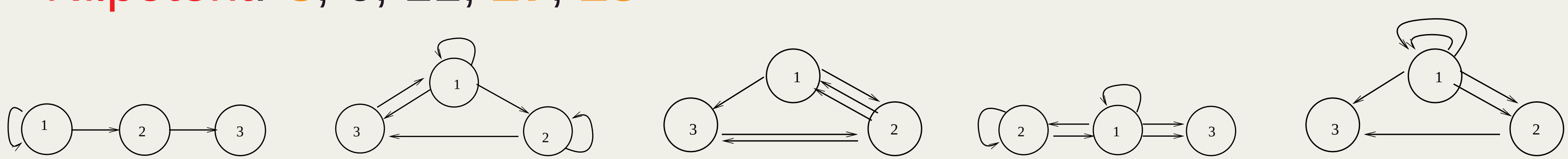
- 20 networks have real simple eigenvalues
- Simple complex eigenvalues: 2, 14, 18, 19, 24



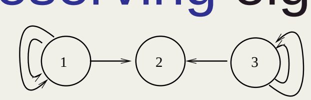
- Double with two synchrony-breaking eigenvectors: 4, 7, 8



- Nilpotent: **3**; 6, 11, **27**, **28**



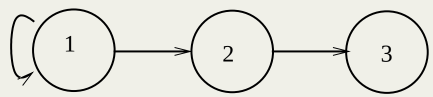
- Double with synchrony preserving eigenvector: 12



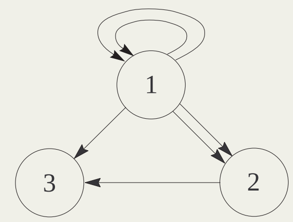
Leite and G. (2006)

Nilpotent Hopf Bifurcation

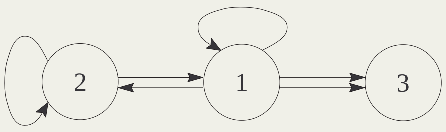
- Networks 3, 28, 27: branches that grow at $\lambda^{\frac{1}{6}}$



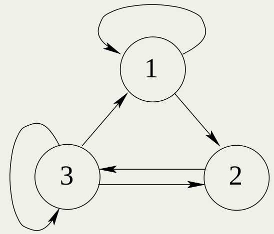
(a)

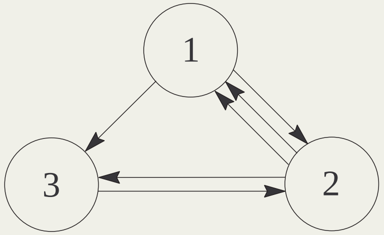


(b)

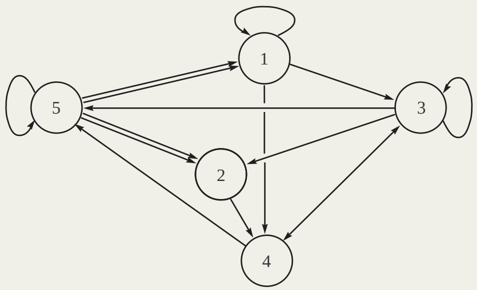


- Networks 6, 11: two or four branches that grow $\lambda^{\frac{1}{2}}$



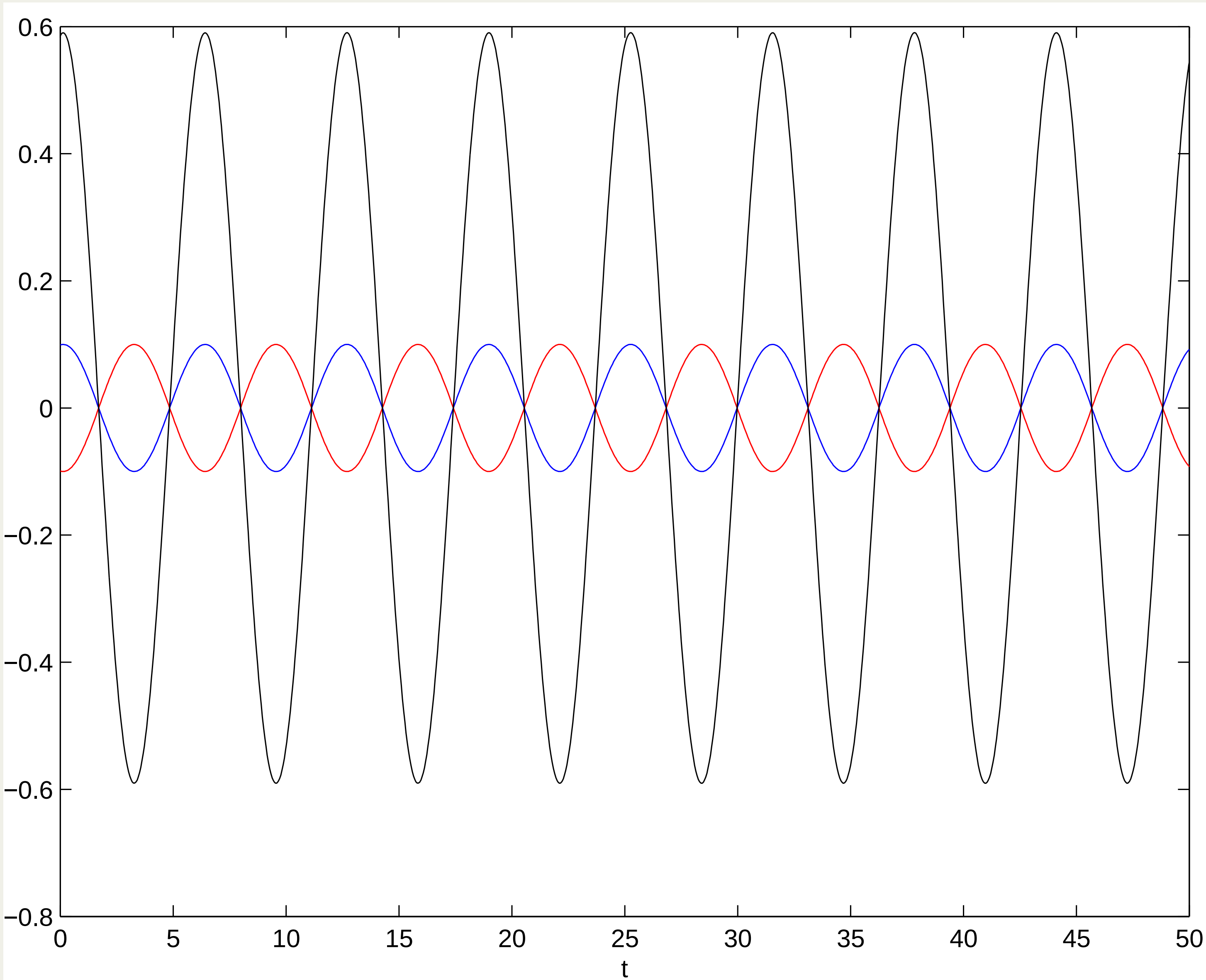


- Regular five-cell network: two branches that grow λ



Elmhirst and G. (2005)

Nilpotent Hopf in Network 27



Thanks

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Andrew Török	Houston	Network Theory
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Yunjiao Wang	MBI	Phase-Shift Synchrony & Lattices
Ana Paula Dias	Porto	Lattices
Fernando Antoneli	Sao Paulo	Lattices